Quasi-Lie algebras

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Abstract. This paper introduces a new class of algebras naturally containing, within a single framework, various deformations of the Witt, Virasoro and other Lie algebras based on twisted and deformed derivations, in addition to color Lie algebras and Lie superalgebras.

1. Introduction

The area of quantum deformations (or $q$-deformations) of Lie algebras began a period of rapid expansion around 1985 when Drinfel’d [Dr] and Jimbo [Ji] independently considered deformations of $U(g)$, the universal enveloping algebra of a Lie algebra $g$, motivated, among other things, by their applications to the Yang–Baxter equation and quantum inverse scattering methods [KR]. Since then several other versions of ($q$-) deformed Lie algebras have appeared, especially in physical contexts such as string theory. The main objects for these deformations were infinite-dimensional algebras, primarily the Heisenberg algebras (oscillator algebras) and the Virasoro algebra, see [CKL, CZ, DK, HLS, HS, Ja] and the references therein. For more details why these algebras are important in physics, see [CKL, CZ, DK, D, DMS, F1, F2], for instance.

An important question with these algebras is whether they obey some deformed (twisted) versions of skew-symmetry or Jacobi identity. At the same time, a well-known direct generalization of Lie algebras and Lie superalgebras to general commutative grading groups is the class of color Lie algebras. In these algebras generalized skew-symmetry and Jacobi type identities, graded by a commutative group and twisted by a scalar bi-character, hold. A remarkable and not yet fully exploited and understood feature of color Lie algebras and Lie superalgebras is that they often appear simultaneously with usual Lie algebras in various deformation families of algebras for initial, final or other important special values of the deformation parameters. The class of algebras we consider in this article is tailored in a...
way suitable for the simultaneous treatment of color Lie algebras and of the deformations arising in connection with twisted, discretized or deformed derivatives.

2. Definitions and notations

We let $\mathbb{F}$ denote a field of characteristic zero and $\mathcal{L}_\mathbb{F}(L)$ the set of linear maps of the linear space $L$ over the field $\mathbb{F}$.

**Definition 2.1.** A quasi-Lie algebra is a tuple $(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega, \theta)$ where

- $L$ is a linear space over $\mathbb{F}$,
- $\langle \cdot, \cdot \rangle_L : L \times L \to L$ is a bilinear map called a product or bracket in $L$;
- $\alpha, \beta : L \to L$, are linear maps,
- $\omega : D_\omega \to \mathcal{L}_\mathbb{F}(L)$ and $\theta : D_\theta \to \mathcal{L}_\mathbb{F}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$,

such that the following conditions hold:

- ($\omega$-symmetry) The product satisfies a generalized skew-symmetry condition $(x,y)_L = \omega(x,y)(y,x)_L$, for all $(x,y) \in D_\omega$;
- (quasi-Jacobi identity) The bracket satisfies a generalized Jacobi identity
  \[ \bigcup_{x,y,z} \{ \theta(z,x) \big( \langle \alpha(x), \langle y,z \rangle_L \rangle_L + \beta(\langle x,y \rangle_L, z) \big) \} = 0, \]
  for all $(x,z), (x,y), (y,z) \in D_\theta$.

Note that $(\omega(x,y)\omega(y,x) - \text{id}) \langle x,y \rangle = 0$, if $(x,y), (y,x) \in D_\omega$, which follows from the computation $(x,y) = \omega(x,y)(y,x) = \omega(x,y)\omega(y,x)(x,y)$.

**Example 2.2.** The class of algebras introduced in Definition 2.1 incorporates as special cases hom-Lie algebras and quasi-hom-Lie algebras (qhl-algebras) (see [HLS, LS, LS^2]) both of which appear naturally in the algebraic study of $\sigma$-derivations (see Theorem 3.1) and related deformations of infinite-dimensional and finite-dimensional Lie algebras. To get the class of qhl-algebras one specifies $\theta = \omega$ and restricts attention to maps $\alpha$ and $\beta$ satisfying the twisting condition $\langle \alpha(x), \alpha(y) \rangle = \beta \circ \alpha(x,y)$. Specifying this further by taking $D_\omega = L \times L$, $\beta = \text{id}$ and $\omega = -\text{id}$, we get the class of hom-Lie algebras including Lie algebras when $\alpha = \text{id}$.

**Example 2.3.** Let $\Gamma$ be an abelian group, and let $L$ be a $\Gamma$-graded linear space over $\mathbb{F}$. A color Lie algebra or $\Gamma$-graded $\varepsilon$-Lie algebra (see [RW, S]) is $\Gamma$-graded linear space $L$ with a bilinear multiplication $\langle \cdot, \cdot \rangle$ satisfying

- $\langle x,y \rangle = -\varepsilon(\gamma_x, \gamma_y) \langle y,x \rangle$
- $\varepsilon(\gamma_x, \gamma_y) \langle x, (y,z) \rangle + \varepsilon(\gamma_x, \gamma_y) \langle y, (x,z) \rangle + \varepsilon(\gamma_y, \gamma_z) \langle z, (x,y) \rangle = 0$

for $x \in L_{\gamma_x}, y \in L_{\gamma_y}$ and $z \in L_{\gamma_z}$. The map $\varepsilon : \Gamma \times \Gamma \to \mathbb{F}$, called a commutation factor, is a bi-character on $\Gamma$ with a symmetry condition, namely a map satisfying

- $\varepsilon(\gamma_x, \gamma_y) \varepsilon(\gamma_y, \gamma_z) = 1$
- $\varepsilon(\gamma_x + \gamma_y, \gamma_z) = \varepsilon(\gamma_x, \gamma_z) \varepsilon(\gamma_y, \gamma_z)$, and $\varepsilon(\gamma_x, \gamma_y + \gamma_z) = \varepsilon(\gamma_x, \gamma_y) \varepsilon(\gamma_x, \gamma_z)$.

Notice that color Lie algebras include both Lie algebras (with $\Gamma = \{ \}$ and $\varepsilon = \text{id}$) and Lie superalgebras (with $\Gamma = \mathbb{Z}_2$ and $\varepsilon(\gamma_x, \gamma_y) = (-1)^{\gamma_x \gamma_y}$). Color Lie algebras are examples of quasi-Lie algebras (in fact they are also qhl-algebras). This can be seen by grading $L$ by $\Gamma$ in the definition of quasi-Lie algebras and putting $\alpha = \beta = \text{id}$ and $\theta(x,y) v = \omega(x,y) v = -\varepsilon(\gamma_x, \gamma_y) v$ for $v \in L$, where $(x,y) \in D_\omega = \cdots$
$D_\theta = \big( \cup_{\gamma \in \Gamma} L_\gamma \big) \times \big( \cup_{\gamma \in \Gamma} L_\gamma \big)$ and $\gamma_x, \gamma_y \in \Gamma$ are the graded degrees of $x$ and $y$. The $\omega$-symmetry and the quasi-Jacobi identity give the respective identities in the definition of a color Lie algebra.

If $L$ as a linear space over $\mathbb{F}$ has a basis $\{e_1, \ldots, e_n\}$ and $(e_j, e_k) \in D_\omega \cap D_\theta$ for any $j, k \in \{1, \ldots, n\}$, then the product is defined completely by its action on the basis

$$\langle e_j, e_k \rangle = \sum_{l=1}^n c_{j,k}^l e_l, \quad c_{j,k}^l \in \mathbb{F} \quad \text{for} \quad j, k, l \in \{1, \ldots, n\},$$

and as in the case of Lie algebras, $\omega$-symmetry and quasi-Jacobi identity can be rewritten in terms of algebraic constraints for the structure constants $\{c_{j,k}^l\}$:

- ($\omega$-symmetry)
  $$\sum_{s=1}^n c_{k,j}^s \omega(e_j, e_k)_{l,s} = c_{j,k}^l$$
- (quasi-Jacobi)
  $$\bigoplus_{j,k,l} \sum_{s,r=1}^n c_{k,j}^s (\alpha_{i,j} c_{i,s}^r + c_{j,s}^r \beta_{r,i}) \theta(e_l, e_j)_{p,r} = 0,$$

for $j, k, l, p \in \{1, \ldots, n\}$, where $\alpha_{i,j}, \beta_{r,i}$, and $\theta(e_l, e_j)_{p,r}$ are matrix elements of the matrices for the twisting operators $\alpha, \beta, \omega(e_j, e_k), \theta(e_l, e_j)$ in the basis $\{e_1, \ldots, e_n\}$. This means that we have a multi-parameter family of varieties. The equations for the varieties of Lie algebras, Lie superalgebras and color Lie algebras are obtained for the special choices of parameters corresponding to the respective choices of twisting maps described in examples 2.2 and 2.3. A very important and nice problem would be to study the (geometric) moduli for these algebras.

**Remark 2.4.** The Jacobi identity for Lie algebras can be rewritten, using skew-symmetry, as the Leibniz rule for the map $x \mapsto \langle x, z \rangle$, namely

$$(\langle x, y \rangle, z) = \langle \langle x, z \rangle, y \rangle + \langle x, \langle y, z \rangle \rangle.$$ (2.1)

Keeping this identity but dropping the requirement that the bracket is skew-symmetric, one gets the class of Leibniz algebras. The similar consideration can be repeated in the case of quasi-Lie algebras. The corresponding form of the quasi-Jacobi identity is

$$\theta(y, z) (\omega(\alpha(z), \langle x, y \rangle)) (\langle x, y \rangle, \alpha(z)) + \beta \circ \omega(z, \langle x, y \rangle) (\langle x, y \rangle, z) =$$

$$= -\theta(x, y) (\omega(\alpha(y), \langle z, x \rangle)) (\omega(z, x) \langle x, z \rangle, \alpha(y)) + \beta \circ \omega(y, \langle z, x \rangle) (\omega(z, x) \langle x, z \rangle, y) -$$

$$- \theta(z, x) (\alpha(x), \langle y, z \rangle) + \beta(x, \langle y, z \rangle)$$

when $(z, x), (x, y), (y, z) \in D_\theta$, $\langle x, y \rangle, \langle z, x \rangle, (y, \langle z, x \rangle), (z, x), (z, x), (z, x) \in D_\omega$. When $\alpha = \beta = \text{id}$ and $\theta = \omega = -\text{id}$, one recovers (2.1) as should be for Lie algebras. So, if we keep the quasi-Jacobi identity in that Leibniz-like form, but drop the $\omega$-symmetry axiom, we get a generalization of Leibniz algebras which may be called quasi-Leibniz algebras.

3. **Quasi-Lie algebras and twisted derivations**

Let $A$ denote an associative algebra over $\mathbb{F}$ with unity 1 and let $\sigma : A \to \text{Z}(A)$ be a fixed but otherwise arbitrary $\mathbb{F}$-algebra homomorphism from $A$ into the center of $A$. A $\sigma$-derivation on $A$ is a $\mathbb{F}$-linear map $D : A \to A$ satisfying the $\sigma$-twisted Leibniz rule

$$D(ab) = D(a)b + \sigma(a)D(b).$$
Denote by $\mathfrak{D}_\sigma(A)$ the set of $\sigma$-derivations on $A$. Fixing an element $\Delta \in \mathfrak{D}_\sigma(A)$ we assume that

\[ \sigma(\text{Ann}(\Delta)) \subseteq \text{Ann}(\Delta), \]

and that there is an element $\delta \in Z(A)$ such that

\[ \Delta(\sigma(a)) = \delta \cdot \sigma(\Delta(a)), \quad \text{for } a \in A, \]

where $\text{Ann}(\Delta) = \{ a \in A \mid a \cdot \Delta = 0 \}$. Let $A \cdot \Delta = \{ a \cdot \Delta \mid a \in A \}$ denote the cyclic $A$-submodule of $\mathfrak{D}_\sigma(A)$ generated by $\Delta$ and extend $\sigma$ to $A \cdot \Delta$ by $\sigma(a \cdot \Delta) = \sigma(a) \cdot \Delta$. In the following key theorem we describe a natural twisted bracket on $A \cdot \Delta$ satisfying a twisted six-term Jacobi identity, making $A \cdot \Delta$ into quasi-Lie algebra.

**Theorem 3.1.** If (3.1) holds then the map $\langle \cdot , \cdot \rangle_\sigma$ defined by setting

\[ \langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a) \cdot \Delta)(b \cdot \Delta) - (\sigma(b) \cdot \Delta)(a \cdot \Delta), \quad \text{for } a, b \in A, \]

where $\circ$ denotes composition of maps, is a well-defined $\mathbb{F}$-algebra product on the $\mathbb{F}$-linear space $A \cdot \Delta$. It satisfies the following identities for $a, b, c \in A$:

\[ \langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta, \]

\[ \langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = -\langle b \cdot \Delta, a \cdot \Delta \rangle_\sigma, \]

and if moreover (3.2) holds, the deformed six-term Jacobi identity

\[ \mathcal{O}_{a,b,c} \left( (\sigma(a) \cdot \Delta)(b \cdot \Delta, c \cdot \Delta) \right)_\sigma + \delta \cdot \langle a \cdot \Delta, (b \cdot \Delta, c \cdot \Delta) \rangle_\sigma = 0. \]

**Proof.** First of all, skew-symmetry and bilinearity are obvious from the definition (3.3). The bracket is well-defined by the assumption (3.1) on the annihilator of $\Delta$. Closure of $\langle \cdot , \cdot \rangle_\sigma$ follows from

\[ \langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma(c) = (\sigma(a) \cdot \Delta)((b \cdot \Delta)(c)) - (\sigma(b) \cdot \Delta)((a \cdot \Delta)(c)) = \]

\[ = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta(c) + (\sigma(a)\sigma(b) - \sigma(b)\sigma(a)) \cdot \Delta(\Delta(c)) \]

since $(\sigma(a)\sigma(b) - \sigma(b)\sigma(a)) \cdot \Delta(\Delta(c)) = 0$ by the assumption $\sigma(A) \subseteq Z(A)$. Let us now prove the deformed Jacobi identity (3.6). This follows, from the assumption $\sigma(A) \subseteq Z(A)$, the relation (3.2) and the fact that $\Delta$ is a $\sigma$-derivation, by the following cyclic summations:

\[ \mathcal{O}_{a,b,c} \langle \sigma(a) \cdot \Delta, (b \cdot \Delta, c \cdot \Delta) \rangle_\sigma = \]

\[ = \mathcal{O}_{a,b,c} \langle \sigma(a) \cdot \Delta, \sigma(b)\Delta(c) - \sigma(c)\Delta(b) \cdot \Delta \rangle_\sigma = \]

\[ = \mathcal{O}_{a,b,c} (\sigma^2(a)\sigma^2(b)\Delta^2(c) - \sigma^2(a)\sigma^2(c)\Delta^2(b)) \cdot \Delta + \]

\[ + \mathcal{O}_{a,b,c} (\sigma^2(c)\sigma(\Delta(b))\Delta(\sigma(a)) - \sigma^2(b)\sigma(\Delta(c))\Delta(\sigma(a))) \cdot \Delta + \]

\[ + \mathcal{O}_{a,b,c} (\sigma^2(a)\Delta(\sigma(b))\Delta(c) - \sigma^2(a)\Delta(\sigma(c))\Delta(b)) \cdot \Delta = \]

\[ = \mathcal{O}_{a,b,c} (\sigma^2(a)\Delta(\sigma(b))\Delta(c) - \sigma^2(a)\Delta(\sigma(c))\Delta(b)) \cdot \Delta. \]

The first two terms vanished when adding up cyclically. Re-write equation (3.4) as $\langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma = (\Delta(c)\sigma(b) - \Delta(b)\sigma(c)) \cdot \Delta$ which is possible since $\sigma(A) \subseteq Z(A)$. 

Now the second part of (3.6) becomes with the aid of (3.2):

\[
\delta \cdot \langle a \cdot \Delta, (b \cdot \Delta, c \cdot \Delta)_\sigma \rangle_

= \mathcal{O}_{a,b,c} \delta \cdot \langle a \cdot \Delta, (\Delta(c)\sigma(b) - \Delta(b)\sigma(c)) \cdot \Delta \rangle_

= \mathcal{O}_{a,b,c} (\delta \cdot \sigma(a)\Delta^2(c)\sigma(b) - \delta \cdot \sigma(a)\Delta^2(b)\sigma(c)) \cdot \Delta +

+ \mathcal{O}_{a,b,c} ( - \Delta(\sigma(c))\sigma^2(b)\Delta(a) + \Delta(\sigma(b))\sigma^2(c)\Delta(a)) \cdot \Delta =

= \mathcal{O}_{a,b,c} ( - \Delta(\sigma(c))\sigma^2(b)\Delta(a) + \Delta(\sigma(b))\sigma^2(c)\Delta(a)) \cdot \Delta,
\]

where the first of the two terms in the sum before the last equality sign happens to be zero because of cyclic summation and \(\sigma(A) \subseteq Z(A)\). Combining this with (3.7) yields (3.6).

**Remark 3.2.** Another set of assumptions making the above proof valid is:

1. \(\sigma(A)\) is commutative;
2. \(\Delta(A) \subseteq \text{Centr}(\sigma(A))\) or \(2') \sigma(A) \subseteq \text{Centr}(\Delta(A))\);
3. \(\sigma^2(\cdot) - \sigma \circ \Delta(A) \subseteq \text{Centr}(\delta)\),

where \(\text{Centr}(a) := \{ b \in A \mid ab = ba \}\) is the centralizer of \(a\) and

\[\text{Centr}(C) := \{ b \in A \mid ab = ba \text{ for all } a \in C \subseteq A \}\].

It would be of interest, if at all possible, to find an example where these assumptions hold but \(\sigma(A) \not\subseteq Z(A)\).

**Example 3.3.** Let \(A\) be the unique factorization domain \(\mathbb{F}[t, t^{-1}]\), the Laurent polynomials in \(t\) over the field \(\mathbb{F}\). Then the space \(\mathcal{D}_\sigma(A)\) can be generated by a single element \(D\) as a left \(A\)-module, that is, \(\mathcal{D}_\sigma(A) = A \cdot D\) (Theorem 1 in [HLS]). When \(\sigma(t) = qt\) with \(q \neq 0\) and \(q \neq 1\), one can take \(D\) as \(t\) times the Jackson \(q\)-derivative

\[D = \frac{\text{id} - \frac{\sigma}{1 - q}}{1 - q} : f(t) \mapsto \frac{f(t) - f(qt)}{1 - q}.
\]

Then (3.1) and (3.2) are satisfied with \(\delta = 1\), and Theorem 3.1 yields a quasi-Lie (and even a hom-Lie) algebra structure on \(\mathcal{D}_\sigma(A)\) as follows. The \(\mathbb{F}\)-linear space \(\mathcal{D}_\sigma(A) = \bigoplus_{n \in \mathbb{Z}} \mathbb{F} \cdot d_n\), with \(d_n = -t^n D\) can be equipped with the skew-symmetric bracket \(\langle \cdot, \cdot \rangle_\sigma\) defined on generators by (3.3) as

\[\langle d_n, d_m \rangle_\sigma = q^n d_n d_m - q^m d_m d_n \]

with commutation relations

\[\langle d_n, d_m \rangle_\sigma = \{ n \}_q \{- m \}_q d_{n+m},\]

where \(\{ n \}_q = (q^n - 1)/(q - 1)\) and \(\{ n \}_1 = n\). This bracket satisfies the \(\sigma\)-deformed Jacobi-identity

\[(q^n + 1)\langle d_n, \langle d_t, d_m \rangle_\sigma \rangle_\sigma + (q^t + 1)\langle d_t, \langle d_m, d_n \rangle_\sigma \rangle_\sigma + (q^m + 1)\langle d_m, \langle d_n, d_t \rangle_\sigma \rangle_\sigma = 0.
\]

Obviously this can be viewed as a \(q\)-deformed Witt algebra. It can also be shown that there is a central extension \(\text{Vir}_q\) of this deformation in the category of hom-Lie algebras [HLS], therefore being a natural \(q\)-deformation of the Virasoro algebra. The algebra \(\text{Vir}_q\) is spanned by elements \(\{ d_n \mid n \in \mathbb{Z} \}\) \(\cup \{ c \}\) where \(c\) is central, i.e.,

\[\langle \text{Vir}_q, c \rangle = \langle c, \text{Vir}_q \rangle = 0\].

The bracket of \(d_n, d_m\) computed according to

\[\langle d_n, d_m \rangle = (\{ m \}_q - \{ n \}_q)d_{m+n} + \delta_{m+n,0} \frac{q^m - 1}{6(1 + q^m)} \{ m \}_q \{ m - 1 \}_q c.
\]
Note that when \( q = 1 \) we retain the classical Virasoro algebra.

**Example 3.4.** We let \( \mathcal{A} = \mathbb{F}[t, t^{-1}] \) as before and \( \sigma \) be some non-zero endomorphism such that \( \sigma(t) = qt \). The element \( D = (\text{id} - \sigma)/(\alpha^{-1}, t^k) \) generates a cyclic \( \mathcal{A} \)-submodule \( M \) of \( \mathcal{D}_\sigma(\mathcal{A}) \). The element \( \delta \) can be computed to be \( \delta = q^k t^{(s-1)k} \).

Put \( d_n = -t^n D \). The \( \mathbb{F} \)-linear space \( M = \bigoplus_{i \in \mathbb{Z}} \mathbb{F} \cdot d_i \) allows the structure of an algebra (by Theorem 3.1) with bracket defined on generators (by (3.3)) as

\[
\langle d_n, d_m \rangle_\sigma = q^n d_n s d_m - q^m d_m s d_n
\]

and satisfying relations

\[
\langle d_n, d_m \rangle_\sigma = \alpha q^n d_{ms + n - k} - \alpha q^n d_{ns + m - k},
\]

with \( s \in \mathbb{Z} \) and \( \alpha \in \mathbb{F} \). Note that when \( s \neq 1 \), the expression for the bracket of \( d_n \) and \( d_m \) involves four generators. The \( \sigma \)-deformed Jacobi identity becomes

\[
\bigcirc_{n,m,l} \left( q^n \langle d_{ns}, \langle d_m, d_l \rangle_\sigma \rangle + q^k t^{(s-1)k} \langle d_n, \langle d_m, d_l \rangle_\sigma \rangle \right) = 0.
\]

This defines a quasi-Lie algebra which is also a quasi-hom-Lie algebra but not a hom-Lie algebra. For the detailed computations, see [HLS]. A direct, but tedious computation, shows that the algebra with abstract generators \( \{d_n \mid n \in \mathbb{Z} \} \cup \{t\} \) and relations

\[
q^n d_{ns} d_m - q^n d_{ms} d_n = \alpha q^n d_{ms + n - k} - \alpha q^n d_{ns + m - k},
\]

\[
t \cdot d_n = d_{n+1}, \quad \forall n, m \in \mathbb{Z}
\]

does not satisfy (3.8).

**Example 3.5.** Let \( f \in \mathbb{F}[t] \) and define \( \sigma(f)(t) = f(t+h), h \in \mathbb{F}^* \) and \( \partial_\sigma(f)(t) = \sigma(f)(t) - f(t) = f(t+h) - f(t) \). This operator is easily checked to be a \( \sigma \)-derivation. We have that \( \mathbb{F}[t] \) is a UFD (Unique Factorization Domain) and \( \mathcal{S} := \mathcal{D}_\sigma(\mathbb{F}[t]) = \bigoplus_{n \geq 0} \mathbb{F} \cdot d_n \), where \( d_n = t^n \partial_\sigma \). By Theorem 3.1 we compute the bracket and thus the commutation relations satisfied by the generators \( \{d_k\} \):

\[
\langle d_n, d_m \rangle_\sigma = (t+h)^n \partial_\sigma \circ (t^n \partial_\sigma) - (t+h)^m \partial_\sigma \circ (t^m \partial_\sigma) =
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} h^{n-j} d_j d_m - \sum_{k=0}^{m} \binom{m}{k} h^{m-k} d_k d_n =
\]

\[
= (\sigma(t^n) \partial_\sigma (t^m) - \sigma(t^m) \partial_\sigma (t^n)) \partial_\sigma =
\]

\[
= ((t+h)^n((t+h)^m - t^m) - (t+h)^m((t+h)^n - t^n)) \partial_\sigma =
\]

\[
= ((t+h)^m t^n - (t+h)^n t^m) \partial_\sigma =
\]

\[
= \left( \sum_{j=0}^{m-1} \binom{m}{j} h^{m-j} t^{j+n} - \sum_{i=0}^{n-1} \binom{n}{i} h^{n-i} t^{i+m} \right) \partial_\sigma =
\]

\[
= \left( \sum_{k=n}^{m+n-1} \binom{m}{k-n} h^{m+n-k} k - \sum_{k=1}^{m+n-1} \binom{n}{k-m} h^{n+m-k} k \right) \partial_\sigma =
\]

\[
= \left( \sum_{k=\min(m,n)}^{m+n-1} \binom{m}{k-n} - \binom{n}{k-m} \right) h^{n+m-k} k \partial_\sigma =
\]

\[
= \sum_{k=\min(m,n)}^{m+n-1} \left[ \binom{m}{k-n} - \binom{n}{k-m} \right] h^{n+m-k} d_k,
\]
assuming \( \binom{m}{s} = 0 \) for \( s < 0 \). In the present case we have \( \delta = 1 \), and for \( \alpha = \sigma : a \partial_{\sigma} \mapsto \sigma(a) \partial_{\sigma} \) we note that

\[
\alpha(d_n) = \sigma(t^n) \partial_{\sigma} = \sum_{i=0}^{n} \binom{n}{i} h^i t^{n-i} \partial_{\sigma} = \sum_{i=0}^{n} \binom{n}{i} h^i d_{n-i}.
\]

According to Theorem 3.1, this gives us that \( (\mathcal{S}, \langle \cdot , \cdot \rangle_{\mathcal{S}}, \sigma) \) is a hom-Lie algebra with deformed Jacobi identity

\[
\circ_{n,m,l} \left( \sum_{i=0}^{n} \binom{n}{i} h^i d_{n-i} + d_n, \langle d_m, d_l \rangle \right) = 0.
\]

The algebra thus obtained is a natural Witt-type algebra associated to an additive shift difference operator \( \partial_{\sigma} \), frequently used in many parts of mathematics, science and engineering.

An interesting direction with many potential applications is to construct and study deformed Virasoro-type algebras obtained as central extensions of this hom-Lie algebra, as well as more general quasi-Lie algebras derived from Theorem 3.1, within the classes of hom-Lie algebras, quasi-hom Lie algebras and quasi-Lie algebras. Another possibly fruitful line of investigation would be to study general extensions of one quasi-Lie algebra by another. For a construction in this spirit for the Lie algebra case see [JL] wherein is remarked that a certain short exact sequence can be seen as a central extension of Vir by an infinite-dimensional Heisenberg algebra \( \mathfrak{h}_\infty \).

References


[LS2] D. Larsson and S.D. Silvestrov, Quasi-Deformations of $\mathfrak{sl}_2(F)$ using twisted derivations, math.RA/0506172, to appear
