Burchnall–Chaundy theory, Ore extensions
and $\sigma$-differential operators

Daniel Larsson

Department of Science, Buskerud and Vestfold University College
P.O. Box 235, 3603 Kongsberg, Norway
daniel.larsson@hbv.no

Received 5 November 2012
Accepted 10 February 2014
Published 21 April 2014

Communicated by S. K. Jain

A classical theorem of J. L. Burchnall and T. W. Chaundy shows that two commuting
differential operators $P$ and $Q$ give rise, via a differential resultant, to a complex alge-
braic curve with equation $F(x, y) = 0$, such that formally inserting $P$ and $Q$ for $x$ and
$y$ in $F(x, y)$, gives identically zero. In addition, the points on this curve have coordi-
nates which are exactly the eigenvalues associated with the operators $P$ and $Q$ (see the
Introduction for a more precise statement). In this paper, we prove a generalization of
this result using resultants in Ore extensions.

Keywords: Burchnall–Chaundy theory; Ore extensions; $\sigma$-differential operators; alge-
braic curves.

Mathematics Subject Classification: 16U20, 39A13, 12H05, 12H10, 14H50

Notations.

The following notations will be adhered to throughout.

- By $k$ we denote a commutative ground ring, most often a field;
- All algebras, unless explicitly specified otherwise, will be over $k$ and commutative,
  associative with unity;
- In addition, all algebras will be integral domains;
- $A^\times$ denotes the set of units in $A$;
- $t$ and $z$ always denote transcendental indeterminates, i.e. they are not algebraic
  over the given base ring;
- $\partial$ will denote derivation with respect to $t$, i.e. $\partial := d/dt$;
- “Geometric” variables, i.e. generators of the coordinate ring defining algebraic
  varieties, are written in boldface letters $x, y$;
- All difference and differential operators are assumed to be linear over $k$. 
D. Larsson

1. Introduction

In the 1920s, Burchnall and Chaundy discovered in a series of papers, beginning with [1], a remarkable connection between complex algebraic curves and pairs of commuting differential operators. They found that given two operators

\[ P := \sum_{i=0}^{n} p_i \partial^i \quad \text{and} \quad Q := \sum_{i=0}^{m} q_i \partial^i, \]

such that \( PQ - QP = 0, \) and where \( p_i, q_i \) are complex-valued functions, there is a canonical, and explicitly computable, complex algebraic curve \( \mathcal{C} \) with equation \( F(x, y) \in \mathbb{C}[x, y] \) such that \( F(P, Q) = 0. \) This curve is computable via a differential resultant, i.e. the determinant of the matrix formed by the coefficients in

\[ \partial^k (P - x), \quad k = 0, 1, \ldots, n - 1, \quad \partial^\ell (Q - y), \quad \ell = 0, 1, \ldots, m - 1. \]

It is then a fact that the power of \( t \) in all terms of the expanded determinant are the same and can thus be factored out, leaving a polynomial in \( x, y \) with complex coefficients annihilating the operators \( P \) and \( Q. \)

Moreover, the points \((x, y)\) on the curve \( \mathcal{C} \) are exactly the eigenvalues of the joint eigenproblem \( P\psi = x\psi \) and \( Q\psi = y\psi. \) This defines a vector bundle over \( \mathcal{C} \) with sections being the eigenfunctions.

Burchnall and Chaundy’s discovery was forgotten until the 1970s when Krichever rediscovered their results in a series of papers concerned with integrable systems. Since then this theory has become somewhat of an industry. Previato’s survey [10] might be a bit outdated but is still a rather good place to start exploring the subject. Another nice, rather gentle, introduction is provided by Mulase [8]. For a more up to date state of the theory, we urge the reader to simply make an internet search and will thereby find numerous papers. The algebraic-geometrically inclined reader might enjoy Mumford’s treatment [9].

In [3], was proved an analogous theorem of Burchnall and Chaundy for \( q \)-difference operators (in Mumford’s paper there is a section on additive difference operators). This \( q \)-analog showed the existence and presented an algorithm for constructing an annihilating curve. In [5], an exact analogue of the resultant scheme of Burchnall and Chaundy was proposed and proved in a series of examples to yield a correct annihilating curve. No general proof was given to the effect of showing that this construction works in all cases.

This problem was addressed in [2] where a proof was given that this \( q \)-resultant scheme actually works to produce such an annihilating curve of two commuting \( q \)-difference operators. To be precise, they produce a family of algebraic curves annihilating the two given operators. Their argument is, for the most part, analytical and (in my opinion) rather involved.

In this paper, we use a result of Li [6] to prove a much more general result. In addition to this, the proof is much simpler (in fact, it is an easy application of Li’s result) than the one given in [2] in the special case of \( q \)-difference operators.
The first thing to note is that $q$-difference operators are special cases of so-called $\sigma$-differential operators, built from $\sigma$-twisted derivations in the same way differential operators are built from derivations. A $\sigma$-twisted derivation, or $\sigma$-derivation for short, is a $k$-linear map $\partial_\sigma$ on a $k$-algebra $A$ such that

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b),$$

where $\sigma$ is a $k$-algebra endomorphism. In the case of $q$-difference operators on an algebra of functions (say, in one variable $t$, over a field), we have the following two examples with $\sigma(f(t)) = f(qt)$

$$\Delta_\sigma(f(t)) = \sigma(f(t)) - f(t) = f(qt) - f(t), \quad \text{and}$$

$$\partial_\sigma(f(t)) = \frac{\sigma(f(t)) - f(t)}{\sigma(t) - t} = \frac{f(qt) - f(t)}{(q-1)t}.$$

A $\sigma$-differential operator is an operator on the form:

$$P = \sum_{i=0}^{n} p_i \partial_\sigma^i,$$

where $p_i \in A$,

Second, we use a representation of these twisted operators as elements of Ore extension rings (skew-polynomial rings). Given two commuting skew-polynomials representing two commuting twisted operators $P$ and $Q$, the result of Li [6] is used to produce a family of commutative polynomials in two indeterminates (over subrings of the ring $A$ generated by the coefficients of $P$ and $Q$) annihilating $P$ and $Q$.

In Sec. 2. we recall the definition of Ore extensions and introduce some handy notation. In addition, we discuss difference equations and $\sigma$-differential operators and how to represent these as elements in Ore extensions. We also prove some elementary results that will be needed later. Then in Sec. 3 we review the relevant parts of Li’s work [6]. So, in Sec. 4 we prove the main results, one of which is simply a two-line application of the discussion in Sec. 3. Finally, Sec. 5 is devoted to examples and discussion of possible directions for future investigations.

The present paper is, to a large part, self-contained.
and \( \pi^k_0 = \partial^k_0 \). We also put \( \pi^n_k = 0 \) for \( n < k \) and \( k < 0 \). The lemma and proposition below can be found in [4].

**Lemma 2.1.** \( \pi^{n+1}_k = \partial_n \circ \pi^n_k + \sigma \circ \pi^n_{k-1} \).

**Proof.** Simple induction.

**Proposition 2.2.** The following holds on an algebra \( A \) (not necessarily commutative)

(i) \( \pi^n_k(ab) = \sum_{i=k}^n \pi^n_i(a)\pi^n_k(b) \) for \( i \leq n \) and \( a, b \in A \).

(ii) \( \partial^n_k(ab) = \sum_{i=0}^n \pi^n_i(a)\partial^n_k(b) \) (Leibniz’ rule for \( \sigma \)-derivations).

**Proof.** (i) Follows by a straightforward, but somewhat messy, induction on \( n \) using the above Lemma, and (ii) follows from (i) by taking \( k = 0 \). An alternative, and much simpler, way to prove (i) is indicated in [4].

We are interested in the Ore extension \( A[z; \sigma, \partial_\sigma] \). Recall that this is a twisted polynomial algebra over \( A \) satisfying the commutation relation

\[
za = \sigma(a)z + \partial_\sigma(a).
\]

It is rather easy to see that

\[
z^a := \sum_{i=0}^n \pi^n_i(a)z^i,
\]

where \( \pi^n_i \) is the combinatorial object given above and that this endows \( A[z; \sigma, \partial_\sigma] \) with the structure of \((\mathbb{N}_0\text{-graded}) A\)-algebra (see e.g. [7] for more details). We let \( A[z; \sigma, \partial_\sigma] \) act on the \( A \)-module \( M \) by the rule

\[
az^i(m) := a \cdot \partial^i_\sigma(m).
\]

Observe the special case \( M = A \), which is in fact the one we are most interested in.

**Lemma 2.3.** This action is well-defined, that is, the action is associative

\[
az^i(bz^j(m)) = (az^ibz^j)(m).
\]

**Proof.** The proof follows easily by induction and the associativity of \( \partial^n_k \) as follows.

We have for \( j \geq 0 \),

\[
az^0(bz^j(m)) = abz^j(m) = ((ab) \cdot \partial^j_\sigma)(m) = (a \cdot \partial^0_\sigma b \cdot \partial^j_\sigma)(m) = (az^0bz^j)(m).
\]

Assume now that \( i > 0 \) and that the result holds for \( i - 1 \). Then

\[
az^i(bz^j(m)) = az^{i-1}(\sigma(b)z^{j+1} + \partial_\sigma(b)z^j)(m)
\]

\[
= az^{i-1}(\sigma(b)z^{j+1}(m) + az^{i-1}\partial_\sigma(b)z^j(m))
\]

\[
= (az^{i-1}\sigma(b)z^{j+1})(m) + (az^{i-1}\partial_\sigma(b)z^j)(m)
\]

\[
= (az^i bz^j)(m).
\]

Extending linearly proves the claim.
To simplify we will denote $A[z; \sigma, \partial_{\sigma}]$ simply as $A\{z\}$. Elements in $A\{z\}$ are often referred to as Ore polynomials or skew polynomials.

### 2.1. $\sigma$-differential operators and difference modules

#### 2.1.1. $\sigma$-differential operators

Recall that all algebras are assumed to be integral domains.

A $\sigma$-differential operator is an operator on $M$ on the form

$$ P := \sum_{i=0}^{n} p_i \sigma_i^x, \quad p_i \in A, $$

and a $\sigma$-differential equation is then an equation on the form

$$ P\psi = \sum_{i=0}^{n} p_i \sigma_i^x \psi = 0, \quad \text{where } p_i \in A, \text{ and } \psi \in M. $$

#### 2.1.2. Difference modules

There is another, closely related, and in fact in many cases equivalent, formulation of $\sigma$-differential equations.

**Definition 2.1.** We make the following definitions.

- A $\sigma$-difference ring is a ring $A$ together with a $\sigma \in \text{End}(A)$.
- The difference ring $(A, \sigma)$ is called simple if the only $\sigma$-invariant proper ideal is $(0)$.
- A $(\Phi, \sigma)$-difference module $(M, \Phi)$ is an $A$-free module over a difference ring $(A, \sigma)$ together with a $\sigma$-linear endomorphism $\Phi$.
- A $\sigma$-difference equation is an equation on the form

$$ \sum_{i=0}^{n} s_i \sigma_i^x \psi = 0, \quad \text{where } \psi, s_i, \text{ are elements in } A. $$

To recall, the notion $\sigma$-linear means that

$$ \Phi(am) = \sigma(a)\Phi(m), \quad \text{for } a \in A, \ m \in M. $$

**Remark 2.1.** It is strictly not necessary to assume that a difference module is $A$-free. In the most general case that I know of, it is only assumed that the module is $A$-flat.

To every difference equation can be associated a difference module and conversely. Notice first that, just as in the case of differential equations, a difference
Lemma 2.5. We have
\[
\sigma^n = \sum_{i=0}^{n-1} \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i}.
\]

Proof. Every operator \( P = \sum_{i=1}^{n} p_i \sigma^i \) can clearly be written as a linear combination of powers of \( \sigma \) after expanding the powers of \( \text{id} - \sigma \) and rearranging. This allows us to rewrite \( P \) uniquely as \( P = \sum_{i=1}^{n} c_i \sigma^i \), where \( c_i := \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i} \).

Proposition 2.4. Every \( \sigma \)-difference operator \( \sum_{i} a_i \sigma^i \) over a \( \sigma \)-difference ring \( \left( A, \sigma \right) \) can be uniquely expressed as a \( \sigma \)-differential operator \( \sum_{i} c_i \partial_i \sigma \), where \( \partial_i \sigma := \text{id} - \sigma \).

Proof. Every operator \( P = \sum_{i=1}^{n} p_i \partial_i \sigma \) can clearly be written as a linear combination of \( \sigma \) after expanding the powers of \( \text{id} - \sigma \) and rearranging. This allows us to rewrite \( P \) uniquely as \( P = \sum_{i=1}^{n} c_i \sigma^i \), where \( c_i := \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i} \).

On the other hand, given a difference module \( \left( M, \Phi \right) \), we can always turn this into a difference equation by \( \sigma^i X = S^{-1} \sigma^i X \), where \( S \) is invertible. That \( S \) is invertible is equivalent to \( s_0 = 0 \). A solution to (2.1) lies in \( \ker(\Phi - \text{id}) \), where \( \Phi := S^{-1} \sigma \). Conversely, any element in \( \ker(\Phi - \text{id}) \) is a solution to (2.1).

Proposition 2.5. Every \( \sigma \)-difference operator \( \sum_{i} a_i \sigma^i \) over a \( \sigma \)-difference ring \( \left( A, \sigma \right) \) can be uniquely expressed as a \( \sigma \)-differential operator \( \sum_{i} c_i \partial_i \sigma \), where \( \partial_i \sigma := \text{id} - \sigma \).

Proof. Every operator \( P = \sum_{i=1}^{n} p_i \partial_i \sigma \) can clearly be written as a linear combination of \( \sigma \) after expanding the powers of \( \text{id} - \sigma \) and rearranging. This allows us to rewrite \( P \) uniquely as \( P = \sum_{i=1}^{n} c_i \sigma^i \), where \( c_i := \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i} \).

On the other hand, given a difference module \( \left( M, \Phi \right) \), we can always turn this into a difference equation by \( \sigma^i X = S^{-1} \sigma^i X \), where \( S \) is invertible. That \( S \) is invertible is equivalent to \( s_0 = 0 \). A solution to (2.1) lies in \( \ker(\Phi - \text{id}) \), where \( \Phi := S^{-1} \sigma \). Conversely, any element in \( \ker(\Phi - \text{id}) \) is a solution to (2.1).

Proof. Every operator \( P = \sum_{i=1}^{n} p_i \partial_i \sigma \) can clearly be written as a linear combination of \( \sigma \) after expanding the powers of \( \text{id} - \sigma \) and rearranging. This allows us to rewrite \( P \) uniquely as \( P = \sum_{i=1}^{n} c_i \sigma^i \), where \( c_i := \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i} \).

On the other hand, given a difference module \( \left( M, \Phi \right) \), we can always turn this into a difference equation by \( \sigma^i X = S^{-1} \sigma^i X \), where \( S \) is invertible. That \( S \) is invertible is equivalent to \( s_0 = 0 \). A solution to (2.1) lies in \( \ker(\Phi - \text{id}) \), where \( \Phi := S^{-1} \sigma \). Conversely, any element in \( \ker(\Phi - \text{id}) \) is a solution to (2.1).

Proof. Every operator \( P = \sum_{i=1}^{n} p_i \partial_i \sigma \) can clearly be written as a linear combination of \( \sigma \) after expanding the powers of \( \text{id} - \sigma \) and rearranging. This allows us to rewrite \( P \) uniquely as \( P = \sum_{i=1}^{n} c_i \sigma^i \), where \( c_i := \left( \sigma^i \right) \left( \text{id} - \sigma \right)^i \left( \text{id} - \sigma \right)^{n-i} \).

On the other hand, given a difference module \( \left( M, \Phi \right) \), we can always turn this into a difference equation by \( \sigma^i X = S^{-1} \sigma^i X \), where \( S \) is invertible. That \( S \) is invertible is equivalent to \( s_0 = 0 \). A solution to (2.1) lies in \( \ker(\Phi - \text{id}) \), where \( \Phi := S^{-1} \sigma \). Conversely, any element in \( \ker(\Phi - \text{id}) \) is a solution to (2.1).
First of all, using the binomial identity \((\text{id} - \sigma)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sigma^i\), it is obvious that we can write \(\sigma^n = \sum_{i=0}^{n} a_i (\text{id} - \sigma)^i\) for some (unique) \(a_i \in \mathbb{Z}\) after some rearrangement, and this is actually sufficient to conclude the last statement of the above proposition. But to prove the explicit form given by the lemma we proceed as follows.

We will use the following combinatorial identities:

(a) \(\binom{n}{i} \binom{i}{j} = \binom{n}{j} \binom{n-j}{i-j}\);

(b) \(\sum_{i=0}^{k} (-1)^i \binom{n}{i} = (-1)^k \binom{n-1}{k}\).

It is easily seen that formula (2.2) holds for \(n = 2\) and \(n = 3\). For induction assume that it holds for all \(i < n\).

From the binomial identity follows that

\[
\sigma^n = (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \sigma^i + (-1)^n (\text{id} - \sigma)^n.
\]

To simplify notation we use \(\alpha := \text{id} - \sigma\). Using the induction hypothesis we have

\[
\sigma^n = (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \sigma^i + (-1)^n \alpha^n
\]

\[
= (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \alpha^j + (-1)^n \alpha^n
\]

\[
= (-1)^{n+1} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} \alpha^j + (-1)^n \alpha^n
\]

After some rearranging this can be written as

\[
\sigma^n = (-1)^{n+1} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} \binom{n}{j} \binom{j}{i} \alpha^j + (-1)^n \alpha^n
\]

By identity (a) above this can be written as

\[
\sigma^n = (-1)^{n+1} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j} \binom{n}{j} \binom{j-i}{i} \alpha^i + (-1)^n \alpha^n
\]

By shifting indices in the innermost sum, putting \(l = j - i\), that sum can be computed as

\[
\sum_{l=0}^{n-i-1} (-1)^{l+i} \binom{n-i}{l} = (-1)^i \sum_{l=0}^{n-i-1} (-1)^l \binom{n-i}{l}
\]
and using (b) this can be written as
\[
(-1)^i \sum_{l=0}^{n-i-1} (-1)^l \binom{n-i}{l} = (-1)^{n-i} \binom{n-i-1}{n-i-1} = (-1)^{n-1}.
\]
Hence,
\[
\sigma^n = (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (-1)^{n-i-1} \sum_{j=i}(-1)^j \binom{n-i-1}{j-i} \alpha^i + (-1)^{n+1}
\]
\[
= (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \alpha^i + (-1)^{n+1} \alpha^n
\]
\[
= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \alpha^i + (-1)^{n+1} \alpha^n
\]
proving the lemma.

Now, let \((E, \Phi)\) be a \(\sigma\)-difference module over a \(\sigma\)-difference ring \((A, \sigma)\). Furthermore, we put \(\partial_\sigma \defeq t(id - \sigma)\) for \(t \in A\). Consider the map
\[
\nabla(\sigma) : E \rightarrow (A \cdot \varepsilon) \otimes_A E, \quad m \mapsto \varepsilon \otimes t(id - \Phi)(m), \quad t \in A,
\]
\[(2.3)\]
where \(A \cdot \varepsilon\) is the canonical rank one \(A(U)\)-module with basis \(\varepsilon\). This map satisfies a twisted Leibniz rule:
\[
\nabla(\sigma)(am) = \varepsilon \otimes (t(id - \sigma)(a)m) + \varepsilon \otimes (a^\sigma t(id - \Phi)(m))
\]
\[
= (t(id - \sigma)(a) \cdot \varepsilon) \otimes e + a^\sigma \cdot \varepsilon \otimes t(id - \Phi)(m),
\]
i.e.
\[
\nabla(\sigma)(am) = \partial_\sigma(a) \cdot \varepsilon \otimes m + a^\sigma \nabla(\sigma)m.
\]
In addition, we can easily see that
\[
\nabla(\sigma) \circ \Phi = q \cdot \Phi \circ \nabla(\sigma),
\]
where \(q = \sigma(t)/t\).

Lemma 2.6. Keeping the notation from above, there is a canonical \(\sigma\)-twisted connection \(\nabla_M^{(\sigma)}\) given by (2.3). Conversely, localizing at \(t\) if necessary, given a \(\sigma\)-twisted connection we have a canonical \(\sigma\)-difference module \((M_{\nabla}, \Phi)\) as the kernel of \(\nabla^{(\sigma)}\).

We see that we can either view a difference equation as a difference module or, equivalently, as a twisted connection.

Remark 2.2. By twisting the actions of \(A\) on \(A \cdot \varepsilon\) by the rule
\[
\alpha \varepsilon = \varepsilon \alpha^\sigma
\]
we can in fact transform the above twisted Leibniz rules into “ordinary” Leibniz rules:

$$\nabla^{(\sigma)}(am) = \partial_\tau(a) \cdot \varepsilon \otimes m + a \nabla^{(\sigma)} m.$$

This actually means that $A \cdot \varepsilon$ is the first-order part of the twisted polynomial ring $A\{t, \sigma\}$, with $a \cdot \varepsilon = \varepsilon a^\sigma$. Notice that this is in fact the twisted polynomial ring $A\{T, \sigma\}$, with $aT = T \sigma(a)$.

3. Resultants in Ore Extensions

We will now review the relevant parts of [6].

Assume now that $A$ is a commutative $k$-algebra. Let $m$ be an $(r \times c)$-matrix, $r \leq c$.

$$m := \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1,r-1} & m_{1r} & \cdots & m_{1c} \\
m_{21} & m_{22} & \cdots & m_{2,r-1} & m_{2r} & \cdots & m_{2c} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
m_{r1} & m_{r2} & \cdots & m_{r,r-1} & m_{rr} & \cdots & m_{rc}
\end{pmatrix},$$

with all $m_{ij} \in A$.

The determinant polynomial of $m$ is defined as

$$P(m) := \sum_{i=0}^{c-r} \det(m_i) z^i \in A[z],$$

where $m_i$ is the matrix whose first $(r - 1)$-columns are the ones from $m$ and the $r$th is the $(c - i)$th from $m$.

Put $J_i := m_{i1} z^{-i} + m_{i2} z^{-i-1} + \cdots + m_{ic} z^{-c} + \cdots + m_{ir} z^{-r+i-1} + m_{ir} z^{-r+i}$, $i = 1, \ldots, r$. Then the determinant polynomial can be written as

$$P(m) = \det \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1,r-1} & J_1 \\
m_{21} & m_{22} & \cdots & m_{2,r-1} & J_2 \\
\vdots & \vdots & & \vdots & \vdots \\
m_{r1} & m_{r2} & \cdots & m_{r,r-1} & J_r
\end{pmatrix}. \quad (3.1)$$

We use the convention that elements not involving $z$ are always multiplied from the left, for instance, $\text{mult}(az^i, b) := b a z^i$. The reason for this explicit convention is that $a \in A$ and $z$ does not commute. With this convention, expanding the determinant (3.1) is non-ambiguous.

Let $F := (F_1, F_2, \ldots, F_r)$ be a sequence of skew-polynomials and put

$$d := \max\text{deg}(F) + 1.$$

Form the matrix $m(F)$ with $(i, j)$th entry the coefficient of $z^{d-j}$ in polynomial $F_i$, for $1 \leq i \leq r$, $1 \leq j \leq d$. The determinant polynomial of $F$,

$$\text{DetPol}(F) = \text{DetPol}(F_1, F_2, \ldots, F_r),$$

is defined as the determinant polynomial $\text{DetPol}(m(F))$. 

1450049-9
D. Larsson

**Definition 3.1.** For \( P, Q \in A\{z\} \), with \( n := \deg(P) \) and \( m := \deg(Q) \), \( n \geq m \), we define the \( \ell \)th subresultant of \( P \) and \( Q \) as

\[
sres_{\ell}(P, Q) := \text{DetPol}(z^{m-\ell-1}P, \ldots, zP, P, z^{n-\ell-1}Q, \ldots, Q, Q),
\]

where \( \ell = 0, 1, \ldots, m - 1 \). The \( m \)th subresultant is defined simply as \( Q \). The subresultant sequence is the sequence of skew-polynomials

\[
s\text{Seq}(P, Q) := \{sres_0(P, Q), sres_1(P, Q), \ldots, sres_{m-1}(P, Q), P, Q\}.
\]

The 0th subresultant \( sres_0(P, Q) \) is called the resultant of \( P \) and \( Q \), denoted \( \text{Res}(P, Q) \).

**4. The Main Result**

**Proposition 4.1.** For \( P \) and \( Q \) skew polynomials, we have for all \( 0 \leq \ell \leq m - 1 \), where \( m \leq n \), \( sres_{\ell}(P, Q) = S_{\ell}P + T_{\ell}Q \), for some skew polynomials \( S_{\ell} \) and \( T_{\ell} \). In particular, \( \text{Res}(P, Q) = S_0P + T_0Q \).

**Proof.** This follows from the definition and (3.1).

If the polynomials we want to investigate are \( P - x \) and \( Q - y \) with \( [P, Q] = PQ - QP = 0 \) then the resultant is a polynomial in commuting variables \( x \) and \( y \) with coefficients from \( A \) and

\[
R^\bullet(x, y) := sres_\bullet(P - x, Q - y) = S_\bullet(P - x) + T_\bullet(Q - y), \quad \bullet = 0, \ldots, m - 1.
\]

We put \( R(x, y) := R^0(x, y) \). Hence by formally evaluating \( R(x, y) \) in \((P, Q)\) we get zero. Observe that we need that \( P \) and \( Q \) commute since otherwise the polynomial \( R^\bullet(x, y) \) does not make sense as a commutative polynomial.

**Corollary 4.2.** Let \( P \) and \( Q \) commute. Then \( R^\bullet(P, Q) = 0 \), \( \bullet = 0, \ldots, m - 1 \), where \( m = \deg(Q) \leq n = \deg(P) \). In particular, \( R(P, Q) = 0 \).

We denote the curve defined by \( R(x, y) \) by \( BC(P, Q) \) and call it the Burchnall–Chaundy curve, or simply BC-curve, associated with \( P \) and \( Q \). Most often \( P \) and \( Q \) are obvious from the context and we simply write \( BC \).

**Remark 4.1.** Notice that this gives a family of \( m \) annihilating curves, not just one as the classical case. However, notice that the coefficients in \( R^i(x, y) \), \( i \neq 0 \), are potentially not defined over the same ring as \( R(x, y) \) (we may need to extend the ring by the coefficients of the new polynomials).

In order to see that the \( sres_j(P, Q) \) actually becomes the classical resultant in the case \( j = 0 \) we briefly recall how the resultant is constructed classically. Suppose \( \deg(P) = n \) and \( \deg(Q) = m, n \geq m \). The rows in the resultant-matrix of \( P \) and \( Q \) are formed as

\[
z^{m-1}P, \ldots, zP, P, z^{n-1}Q, \ldots, Q, Q^i.
\]
By writing each row in “normal form”, that is, transforming it to a polynomial $a_0 + a_1 z + \cdots + a_r z^r$ we put the $j$th coefficient in this polynomial in the $j$th column. Hence the resulting matrix is one where at the $(i, j)$-position there is the $j$th coefficient of the $i$th polynomial. Note that the matrix is quadratic of type $(n + m) \times (n + m)$.

Going back to the definition of the subresultant as given in Definition 3.1, we see that this definition amounts exactly to the above in the case $\ell = 0$ where the polynomials $F$ are the \{\begin{align*}
&z^{m-\ell-1} P, \ldots, zP, P, z^{n-\ell-1} Q, \ldots, zQ, Q
\end{align*}\}.

The only thing left to convince ourselves that the definitions agree when $\ell = 0$ is to see that for quadratic matrices the determinant polynomial is simply the determinant.

Recall from Proposition 2.4 that there is a bijective correspondence between $\sigma$-difference operators and $\sigma$-differential operators. Analogously to differential modules one can define Picard–Vessiot extensions.

**Definition 4.1.** Let $(M, \Phi)$ be a difference module over a difference field $(K, \sigma)$. A $k$-algebra $D$ is called a Picard–Vessiot ring for $(M, \Phi)$ if

1. there is an automorphism $\Phi$ on $D$ extending the $\sigma$ on $K$;
2. $D$ is a simple difference ring;
3. there is a fundamental matrix $F$ for $(M, \Phi)$, and
4. $D$ is minimal with the properties (i)–(iii).

The solution space to $P \psi = 0$, with $P$ a $\sigma$-differential operator (rewritten as a $\sigma$-difference operator via the construction in Proposition 2.4), is the kernel $\ker(\Phi P - \id)$, where $\Phi P - \id$ is the associated $\sigma$-difference system as in Sec. 2.1.2. A fundamental matrix for $(M, \Phi)$ is a matrix $F$ such that the columns of $F$ is a basis for the solution space of $(M, \Phi)$.

Let $k$ be the field of constants $k := K^\sigma$. If $k$ is algebraically closed, a Picard–Vessiot ring to any $(M, \Phi)$ exists and is unique [11, Proposition 1.9]. Recall that to every difference equation $P \psi = 0$, we can associate a (unique) difference module $(M_P, \Phi_P)$ and conversely.

The following proposition shows that when allowing solutions in a Picard–Vessiot ring $D_P$, the solution space is of finite maximal dimension over $k$.

**Proposition 4.3.** Let $D$ be a Picard–Vessiot ring for $(M, \Phi)$ over $(K, \sigma)$ with algebraically closed field of constants $k$. Then

$$\dim_k(\ker(\Phi - \id)) = \dim_K(M),$$

where $\Phi - \id : D \otimes_K M \to D \otimes_K M$ is the natural extension.

**Proof.** See [11, Theorem 1.32].
D. Larsson

Notice that $K^\sigma = \ker(\partial_\sigma : K \to K)$.

**Proposition 4.4.** Let $P$ and $Q$ be commuting operators over a $k$-field $K$ and assume that $k = \ker(\partial_\sigma : K \to K)$ is algebraically closed. Then a joint eigenvector to

\[
\begin{align*}
P\psi &= x\psi, \\
Q\psi &= y\psi,
\end{align*}
\]

exists, possibly after extending scalars to a Picard–Vessiot ring $D$. The eigenvalues $x, y \in k$ for any joint solution are elements in $\mathcal{BC}(k)$, where the curve $\mathcal{BC}$ is defined by $R(x, y) = 0$. Furthermore, $\mathcal{BC}$ is actually defined over $k$.

If $L(x, y)$ is the vector space of all solutions at $(x, y)$, then this extends to a coherent sheaf $L$ on $\mathcal{BC}$.

**Remark 4.2.** The notation $\mathcal{BC}(B)$, for $B$ a commutative ring, means the $B$-rational points on $\mathcal{BC}$, i.e. the elements of $B$ which are solutions to $R(x, y) = 0$.

Since $k$ is a proper subfield of $K$ (otherwise the problem would be trivial), and $k$ is algebraically closed, $K$ must be transcendental over $k$. A priori, $R(x, y)$ has coefficients in a subalgebra of $K$ generated by the coefficients of $P$ and $Q$. The proposition shows that $R(x, y)$ is actually defined over $k$.

**Proof of Proposition 4.4.** To be guaranteed that a joint solution to $P\psi = x\psi$ and $Q\psi = y\psi$, with $x, y \in k$, exists we need, a priori, to extend $K$ to a Picard–Vessiot ring $D$ big enough to include solutions to both eigenproblems. However, the proof will show that it is sufficient to extend to a Picard–Vessiot ring for either eigenproblem.

Suppose $\psi \neq 0$ is a joint solution. Then

\[0 = R(P, Q)\psi = R(x, y)\psi.\]

Hence $x, y$ are points on $\mathcal{BC}$. This also shows that $\mathcal{BC}$ is defined over $k$.

The existence of a joint solution can be proved as follows. The results of Sec. 2.1.2 show that associated to $(P - x)\psi$ is a difference module $(M, \Phi)$. By the above Proposition 4.3 we can find a Picard–Vessiot ring $D_P$ over which $(P - x)\psi = 0$ has a solution and such that $\dim_k(\ker(\Phi - \text{id})) = \dim_K(M)$. Hence $\ker(\Phi - \text{id})$ has finite $k$-dimension and therefore also $\ker(P - x)$. Suppose $\psi_x$ is such a solution to $(P - x)\psi = 0$. Then,

\[0 = Q(P\psi_x - x\psi_x) = QP\psi_x - xQ\psi_x = PQ\psi_x - xQ\psi_x,\]

and so $Q\psi_x \in \ker(P - x)$, or in other words, $Q$ acts on $\ker(P - x)$. Therefore, $Q$ has an eigenvector $\psi_x$ on $\ker(P - x)$ with eigenvalues in $k$, since any operator on a finite-dimensional vector space over an algebraically closed field has an eigenvector. This proves existence. Furthermore, it proves that $D_P$ is large enough to guarantee $Q\psi = y\psi$ to have solutions on $\ker(P - x)$.

The last statement is clear.
Burchnall–Chaundy theory, Ore extensions and $\sigma$-differential operators

**Question 4.1.** Let $P$ be an ordinary differential operator. Then, by a theorem of Schur (see [8]), the set

$$C(P) := \{ Q \in \text{Diff}(A) \mid [P, Q] = 0 \},$$

where $\text{Diff}(A)$ is the set of all pseudo-differential operators on $A$ (we assume that $A$ is a "sufficiently nice" $C$-algebra), is a commutative subalgebra of $\mathbb{C}((\partial^{-1}))$. Hence, we can associate a scheme $\text{Spec } C(P)$ to $P$. This turns out to be a curve, called the spectral curve of $P$.

Now, the natural question arises, is this possibly the case for $\sigma$-differential operators also? If not, what is the “maximal special case”?

### 5. Examples

Let $k$ be a field. We denote by $k((t))$ the field of formal Laurent series. Every ring considered below will be a subring (or subfield) of this field.

In addition, except for the first example, we will use the “ordinary” representation of operators and write $\partial$ instead of $z$. This will certainly cause no confusion.

**Example 5.1.** In this example $k = \mathbb{C}$. First, it is clear that ordinary differential operators fit into the present framework. Let $A$ be a ring of $C^\infty$-functions in a variable $t$, for instance $\mathbb{C}[[t]]$. As $\sigma$-differential operators, the differential operators are the ones with $\sigma = \text{id}$. From an Ore extension perspective, we have

$$zt := tz + \partial(t),$$

$$\partial := d/dt.$$

Assuming that $\partial(t) = 1$, we get the Weyl–Heisenberg commutation relation and hence, powers of $z$ act as higher-order differential operators. Hence, given

$$P := \sum_{i=0}^{\infty} p_i(t) \partial^i$$

and

$$Q := \sum_{i=0}^{m} q_i(t) \partial^i, \quad \text{where } p_i, q_i \in \mathbb{C}[[t]]$$

such that $[P, Q] = PQ - QP = 0$, there is a polynomial $F(x, y) \in \mathbb{C}[x, y]$ with the property that $F(P, Q) = 0$. From [8] we learned of the following beautiful, and classical (it is given also in [1]), explicit example. Take

$$P := \partial^2 - 2q(t + \epsilon),$$

$$Q := \partial^3 - 3q(t + \epsilon)\partial - \frac{3}{2}q'(x + \epsilon),$$

where $\epsilon$ is chosen such that $P$ becomes regular at $t = 0$. Then one checks that $[P, Q] = 0$ and that

$$Q^2 = P^3 - \frac{g_2}{4} P - \frac{g_3}{4},$$

where $g_2$ and $g_3$ are the second and third coefficients in the Weierstrass elliptic function $\wp$. In other words, $P$ and $Q$ annihilate the elliptic curve $y^2 = x^3 - \frac{g_2}{4} x - \frac{g_3}{4}$.
Example 5.2. Here, we reproduce an example from [5] on commuting q-difference operators. Let $D_q(f)(t) := \frac{(f(qt) - f(t))}{qt - t}$, on $A = \mathbb{C}[[t]]$ (say), the so-called Jackson $q$-derivative. Suppose $P = t^4D_q^4$ and $Q = t^3D_q^3$. First of all it is easy to check that $[P, Q] = 0$. Second, we need to compute the following determinant (we have rearranged the rows a bit):

$$
\begin{vmatrix}
-x & 0 & 0 & 0 & t^4 & 0 & 0 \\
0 & -x & 0 & 0 & \{4\}_q t^3 & t^4 & 0 \\
0 & 0 & -x & 0 & \{4\}_q \{3\}_q q^2 t & q^3 \{2\}_q \{4\}_q q^3 t & t^4 \\
-y & 0 & 0 & t^3 & 0 & 0 & 0 \\
0 & -y & 0 & \{3\}_q q^2 t & t^3 & 0 & 0 \\
0 & 0 & -y & \{3\}_q \{2\}_q q^t & q^2 \{3\}_q q^2 t & t^3 & 0 \\
0 & 0 & 0 & \{3\}_q q^2 - y & q \{3\}_q q^2 \{2\}_q q^t & q^4 \{3\}_q q^2 q^2 t & t^3 \\
\end{vmatrix}
$$

Expanding yields,

$$F(x, y) = y^4 - x^3 - q^3(q^3 + 2q^2 + 2q + 1)x^3 - q(3q^3 + 4q^2 + 3q + 1)xy^2 - (3q^2 + 2q + 1)x^2 y = 0.$$

Notice the remarkable fact that all powers of $t$ in all the terms are the same, so $t$ can be factored from the whole expression, with the coefficient than the above polynomial $F$. A straightforward, but tedious, computation shows that indeed $F(P, Q) = 0$.

We can interpret this example as defining a surface in $\mathbb{A}^3_C = \text{Spec} \mathbb{C}[x, y, q]$, fibered over $\mathbb{C}$ with one-dimensional fibers. One can prove with some effort that except for a finite number, the fibers are all rational (i.e. birationally equivalent to $\mathbb{P}^1$) with three singular points. It is possible to give the explicit values of $q$ over which the fibers become more complicated, but these values are rather complicated algebraic numbers and (I suspect) not so interesting. Notice however, that origo is a singular point in all fibers.

Example 5.3. Now, given two operators of degree two:

$$P = p_0(t) + p_1(t) \partial_x + p_2(t) \partial_y^2,$$

$$Q = q_0(t) + q_1(t) \partial_x + q_2(t) \partial_y^2,$$

(5.1)

we want to compute the following determinant

$$\det \begin{pmatrix} 
p_0 - x & p_1 & p_2 & 0 \\
\pi_0^1(p_0) & \pi_0^1(p_1) + \pi_1^1(p_0) - x & \pi_0^1(p_2) + \pi_1^1(p_1) & \pi_1^1(p_2) \\
q_0 - y & q_1 & q_2 & 0 \\
\pi_0^1(q_0) & \pi_0^1(q_1) + \pi_1^1(q_0) - y & \pi_0^1(q_2) + \pi_1^1(q_1) & \pi_1^1(q_2) \end{pmatrix}.$$  

(5.2)
Expanding yields:

\[
p_2 \pi_1^1(p_2)y^2 + q_2 \pi_1^1(q_2)x^2 - (q_2 \pi_1^1(p_2) + p_2 \pi_1^1(q_2))xy
\]

\[
+ \{q_1(\pi_1^1(q_2) \pi_0^0(p_2) - \pi_1^1(p_2) \pi_0^0(q_2) + \pi_1^1(q_2) \pi_1^1(p_1) - \pi_1^1(p_2) \pi_1^1(q_1))
\]

\[
+ q_2(\pi_1^1(p_2) \pi_0^0(q_1) - \pi_1^1(q_2) \pi_0^0(p_1) + \pi_1^1(p_2) \pi_1^1(q_0) - \pi_1^1(q_2) \pi_1^1(p_0))\}x
\]

\[
+ \{p_1(\pi_1^1(p_2) \pi_0^0(p_1) - \pi_1^1(p_2) \pi_0^0(q_1) + \pi_1^1(p_2) \pi_1^1(q_1) - \pi_1^1(p_2) \pi_1^1(q_1))\}y
\]

\[
+ \{(p_0q_1 - p_1q_0)(\pi_1^1(q_2)(\pi_1^1(q_2) + \pi_1^1(q_1)) - \pi_1^1(q_2)(\pi_1^1(p_2) + \pi_1^1(p_1)))
\]

\[
+ (p_0q_2 - q_0p_2)(\pi_1^1(q_2)(\pi_1^1(p_1) + \pi_1^1(p_0)) - \pi_1^1(p_2)(\pi_1^1(q_1) + \pi_1^1(q_0)))
\]

\[
+ (p_1q_2 - q_1p_2)(\pi_1^1(p_2) \pi_0^0(q_0) - \pi_1^1(q_2) \pi_0^0(p_0))\}\}
\]

To return to our present example we note that due to apparent complexity of the above equation (even for the case \(n = m = 2\)) we feel that it is worthwhile to also give a somewhat different viewpoint. Incidentally, this also proves that operators on the general form (5.1) annihilate the BC-curve given by the above lengthy equation. Thereafter we give an explicit example.

Expanding (5.2) with respect to the last column we can view (5.2) as the difference of two similar determinants:

\[
\pi_1^1(p_2) \det \begin{pmatrix} p_0 - x & p_1 & p_2 \\ q_0 - y & q_1 & q_2 \\ \pi_0^0(q_0) & \pi_1^1(q_0) + \pi_1^1(q_1) - y & \pi_0^0(q_0) + \pi_1^1(q_1) \end{pmatrix}
\]

\[
- \pi_1^1(q_2) \det \begin{pmatrix} p_0 - x & p_1 & p_2 \\ q_0 - y & q_1 & q_2 \\ \pi_0^0(q_0) & \pi_1^1(q_0) + \pi_1^1(q_0) - x & \pi_0^0(q_0) + \pi_1^1(q_0) \end{pmatrix}.
\]  

(5.3)

Expanding the first with respect to the first row (the other one being completely analogous) we get:

\[
\pi_1^1(p_2)((\pi_0^0(q_2) + \pi_1^1(q_1))(q_1(p_0 - x) - p_1(q_0 - y))
\]

\[
+ (\pi_0^0(q_1) + \pi_1^1(q_0))(p_2(q_0 - y) - q_2(p_0 - x)) + \pi_0^0(q_0)(p_1q_2 - p_2q_1)
\]

\[
+ (q_2(p_0 - x) - p_2(q_0 - y))y\}.
\]

Inserting \(x = P\) and \(y = Q\) we get

\[
\pi_1^1(p_2)((\pi_0^0(q_2) + \pi_1^1(q_1))(p_1q_2 - p_2q_1)\partial_x^2
\]

\[
+ (\pi_0^0(q_1) + \pi_1^1(q_0))(p_1q_2 - p_2q_1)\partial_x + \pi_0^0(q_0)(p_1q_2 - p_2q_1)
\]

\[
- (p_1q_2 - p_2q_1)\partial_x(q_0 + q_1\partial_x + q_2\partial_x^2)\}.
\]
In a completely analogous way we expand the second determinant to
\[ \pi_1^1(q_2)((\pi_0^1(p_2) + \pi_1^1(p_1))(p_1q_2 - p_2q_1)\partial_\sigma^2 + (\pi_0^1(p_1) + \pi_1^1(p_0))(p_1q_2 - p_2q_1)\partial_\tau + \pi_0^1(p_0)(p_1q_2 - p_2q_1) - (p_1q_2 - p_2q_1)\partial_\tau p_0 + p_1\partial_\sigma + p_2\partial_\sigma^2). \]

The last term here can be written as (and similarly for the first determinant)
\[ \partial_\sigma(p_0 + p_1\partial_\sigma + p_2\partial_\sigma^2) = \pi_0^1(p_0) + (\pi_1^1(p_0) + \pi_0^1(p_1))\partial_\tau + (\pi_0^1(p_2) + \pi_1^1(p_1))\partial_\sigma^2 + \pi_1^1(p_2)\partial_\sigma^2, \]

Using this, the second determinant can be rewritten as
\[ \pi_1^1(q_2)(p_1q_2 - p_2q_1)((\pi_0^1(p_2) + \pi_1^1(p_1))\partial_\sigma^2 + (\pi_0^1(p_1) + \pi_1^1(p_0))\partial_\tau + \pi_0^1(p_0) - (\pi_1^1(p_0) + \pi_0^1(p_1))\partial_\tau - (\pi_1^1(p_2) + \pi_1^1(p_1))\partial_\sigma^2 - \pi_1^1(p_2)\partial_\sigma^2. \]

Hence, we see that what is left of the first and second determinants above are (respectively)
\[ D_1 = -\pi_1^1(p_2)(p_2q_1 - p_1q_2)\pi_1^1(q_2)\partial_\sigma^3, \]
\[ D_2 = -\pi_1^1(q_2)(p_2q_1 - p_1q_2)\pi_1^1(p_2)\partial_\sigma^3, \]
and these two cancels in (5.3), proving the claim.

Now, to be explicit, we will use the following fact: Let \( P := \prod_{i=1}^{n}(t\partial_\sigma + \alpha_i) \) and \( Q := \prod_{i=1}^{m}(t\partial_\tau + \beta_i) \), with \( \alpha_i, \beta_i \in \mathbb{C} \). Then \( P \) and \( Q \) commute, which is easily proved using an induction argument. The problem appears when one tries to compute in this generality. Therefore, we assume that \( \partial_\sigma(t) = 1 \) in addition to \( n = m = 2 \). This means
\[ \pi_0^1(p_0) = 0, \quad \pi_1^1(q_0) = 0, \]
\[ \pi_0^1(p_1) = \alpha_1 + \alpha_2 + 1, \quad \pi_0^1(q_1) = \beta_1 + \beta_2 + 1, \]
\[ \pi_1^1(p_1) = (\alpha_1 + \alpha_2 + 1)\sigma(t), \quad \pi_1^1(q_1) = (\beta_1 + \beta_2 + 1)\sigma(t), \]
\[ \pi_1^1(p_2) = \pi_0^1(q_2) = \partial_\sigma(\sigma(t))t + \sigma^2(t), \quad \pi_1^1(q_2) = \pi_1^1(q_2) = \sigma(\sigma(t)t). \]

Since \( \pi_1^1(p_2) = \pi_1^1(q_2) \), this can be factored and forgotten. Then the above formula reduces to
\[ p_2xy^2 + q_2x^2 - (p_2 + q_2)xy \]
\[ + ((\alpha_1 + \alpha_2 - \beta_1 - \beta_2)q_1\sigma(t) + q_2(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) + q_2(q_0 - p_0))x \]
\[ + ((\beta_1 + \beta_2 - \alpha_1 - \alpha_2)p_1\sigma(t) + p_2(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) + p_2(p_0 - q_0)y \]
\[ + (p_0q_1 - p_1q_0)(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)\sigma(t) \]
\[ + (p_0q_2 - q_0p_2)(\alpha_1 + \alpha_2 - \beta_1 - \beta_2 + p_0 - q_0). \]

A priori, this curve is defined only over \( k[p_0, p_1, p_2, q_0, q_1, q_2] \subseteq k(t) \) and not over \( k \). However, if \( k = k \) is algebraically closed, Proposition 4.4 shows that \( t \) actually must
vanish, giving a curve $\mathcal{BC}(P,Q)$ over $\bar{k}$ with the property that $R(P,Q) = 0$ and every pairs of eigenvalues to $P\psi = x\psi$ and $Q\psi = y\psi$ lie on $\mathcal{BC}(P,Q)$. This can be seen explicitly from the above equation.

**Example 5.4.** Let $P$ and $Q$ be the (commuting) operators

$$P = (t\partial_\sigma - \alpha_1)(t\partial_\sigma - \alpha_2),$$

$$Q = (t\partial_\sigma - \beta_1)(t\partial_\sigma - \beta_2)(t\partial_\sigma - \beta_3) \in \bar{k}(t)[\partial_\sigma],$$

with $\alpha_i, \beta_j \in \bar{k}$, $\sigma(t) = \varepsilon t$ and $\partial_\sigma(t) = 1$. Here $\varepsilon \in \bar{k}$.

Expanding these and computing $\partial_\sigma P$, $\partial_\sigma^2 P$ and $\partial_\sigma Q$, in order to compute the resultant, involves a considerable effort, but is doable by hand. This gives a $5 \times 5$-matrix whose determinant we seek. However, for this determinant it is advisable to use a computer, and not even this gives much information as the coefficients are gruesome. But, the general form and the vanishing of $t$ can be seen from this determinant expression:

$$\mathcal{BC} : \ t^6 \omega_1 y^2 = t^6 (\omega_2 x^3 + \omega_3 xy + \omega_4 x^2 + \omega_5 x + \omega_6). \quad (5.4)$$

As I said, the actual expressions of the $\omega_k$’s in terms of $\alpha_i$ and $\beta_j$ are very complicated, so I will not give them explicitly. However, we see that $\mathcal{BC}$ is a cubic curve over $\bar{k}$ on generalized Weierstrass form. Generically, this defines an elliptic curve.

We can do the above over any ring $R$ and the result will be an equation of the same type, but this time the coefficients will lie in a subalgebra of $R$ generated by the coefficients of $P$ and $Q$. Also note that if we consider the operators over $R(t)$, where $t$ is transcendental over $R$, the same $t$-vanishing phenomena occurs. This is due to the particular form of $P$ and $Q$ and cannot be considered as general fact unless the field of constants is algebraically closed (at least as far as I can see).

The most complicated explicit example I was able to compute was the following. Let

$$P = t\partial_\sigma(t\partial_\sigma - \alpha) = \varepsilon t^2 \partial_\sigma^2 + (1 - \alpha)t\partial_\sigma,$$

$$Q = (t\partial_\sigma)^3 = \varepsilon^3 t^3 \partial_\sigma^3 + \varepsilon(2 + \varepsilon)t^2 \partial_\sigma^2 + t\partial_\sigma.$$}

The resulting BC-curve has Eq. (5.4). After some computation I arrive at the following coefficients (after multiplying by $\varepsilon^{-9}$):

$$\omega_1 = 1,$$

$$\omega_2 = 1,$$

$$\omega_3 = \varepsilon^{-1}(3\varepsilon^2 + 2\varepsilon\alpha + 3\varepsilon + \alpha - 2),$$

$$\omega_4 = \varepsilon^{-1}(\varepsilon - 1)(\varepsilon + 2)(\alpha - (\varepsilon + 1)^2),$$

$$\omega_5 = \varepsilon^{-1}(2\varepsilon^2\alpha^2 - 2\varepsilon^3 + 4\varepsilon^2\alpha + 2\varepsilon\alpha^2 - 6\varepsilon^2 - 5\varepsilon\alpha - \alpha^2 + 4\varepsilon + 2\alpha),$$

$$\omega_6 = 0.$$

It is quite remarkable that the constant term vanishes.
A natural question is for which \( \varepsilon \) and \( \alpha \)'s this defines an elliptic curve. The answer is surprisingly complicated. View (5.4) as a fibration over the affine plane \( \mathbb{A}^2_{\bar{k}} = \text{Spec}(\bar{k}[\varepsilon, \alpha]) \) (i.e. (5.4) is a four-fold fibered over \( \mathbb{A}^2_{\bar{k}} \) with fibers being cubic curves). Then the locus \( \Omega \) in \( \mathbb{A}^2_{\bar{k}} \) over which the fibers are singular (i.e. not elliptic curves) is an algebraic curve over \( \mathbb{Z} \) of degree 78 in \( \varepsilon \) and 15 in \( \alpha \) with several hundred (!) terms. Over the dense open complement \( U := \mathbb{A}^2_{\bar{k}} \setminus \Omega \) the fibers are elliptic curves. In fact, every point \( (\varepsilon_0, \alpha_0) \) determines a \( \sigma \)-difference operator \( \partial_{\sigma} \) and two commuting \( \sigma \)-differential operators such that these operators are annihilated by the fiber over \( (\varepsilon_0, \alpha_0) \) and every joint eigenproblem with these operators has all their eigenvalues on that fiber. If \( (\varepsilon_0, \alpha_0) \in U \), the fiber is an elliptic curve over \( \bar{k} \).

I suspect that these elliptic curves might be worthy of some study.

Acknowledgment

Thanks are due to Sergei Silvestrov for introducing me to this problem a long time ago in the special case of \( q \)-difference operators. The present paper is the result of him bugging me to find an algebraic proof of the main result of [2]. Hopefully this is algebraic and simple enough for him (although he will berate me for not finding a closed expression for the general Burchnall–Chaundy-curve, something I have come to doubt exists in any meaningful form). Thanks also to the referee for a careful reading of the manuscript and for subsequent corrections and remarks.

References

Burchnall–Chaundy theory, Ore extensions and $\sigma$-differential operators
